

BIHOLOMORPHIC INVARIANTS OF A HYPERBOLIC MANIFOLD AND SOME APPLICATIONS

BY

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ABSTRACT. A biholomorphically invariant real function h_X is defined for a hyperbolic manifold X . Properties of such functions are studied. These properties are applied to prove the following theorem. If a hyperbolic manifold X can be exhausted by biholomorphic images of a strictly pseudoconvex domain $D \subset \mathbb{C}^n$ with $\partial D \in C^3$, then X is biholomorphically equivalent either to D or to the unit ball in $\mathbb{C}^n$. The properties of h_D are also applied to some questions concerning the group of analytical automorphisms of a strictly pseudoconvex domain and to similar questions concerning polyhedra.

Introduction. Let X be a hyperbolic manifold of complex dimension n . Let $\mathcal{K} \subset \mathbb{C}^n$ be a bounded homogeneous domain. We denote by $B(x_0, r)$ the ball in Kobayashi metric (see [9]) of radius $r > 0$ with center at $x_0 \in X$. By $\mathcal{R}$ we denote the set of all r such that there exists a biholomorphic imbedding $F: \mathcal{K} \rightarrow X$, $F(\mathcal{K}) \supset B(x_0, r)$. We set

DEFINITION 0.1.

$$(0.1) \quad h_X(x_0, \mathcal{K}) = \inf_{r \in \mathcal{R}} 1/r.$$

Clearly this function is biholomorphically invariant: namely, if $\Phi: X \rightarrow Y$ is biholomorphic, then $h_X = h_Y \circ \Phi$.

Such functions with the property of being invariants could be defined for any holomorphically invariant metric and any homogeneous domain $\mathcal{K}$. For the Carathéodory metric and $\mathcal{K} = U^n$ the unit polydisk, these functions were defined and studied in [4]. In this article we will consider these functions on strictly pseudoconvex domains. The metric we will consider here is the Kobayashi metric, $\mathcal{K}$ will generally be B^n , the unit ball. We will prove that h_X is nonnegative and continuous, and if $h_X(x_0, \mathcal{K}) = 0$ at a point $x_0 \in X$ where X is a hyperbolic manifold, then $h_X \equiv 0$ and X is biholomorphically equivalent to $\mathcal{K}$.

Let $\{X_k\}$, $1 \leq k < \infty$, be a sequence of subdomains of X .

DEFINITION 0.2. We will say that $\{X_k\}$ exhausts X if for any compact $K \subset X$ there exists a number N such that $X_n \supset K$ for any $n > N$.

One of the properties of h_X is that $h_{X_n} \rightarrow h_X$ uniformly on compacta if X is completely hyperbolic. (A hyperbolic manifold is said to be complete if all subsets bounded in the Kobayashi metric are relatively compact in X .)

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Let $h_D(z) = h_D(z, B^n)$ where D is a bounded strictly pseudoconvex domain with $\partial D \in C^3$. The main property of this function is

$$(0.2) \quad h_D(z) \xrightarrow{z \rightarrow \partial D} 0.$$

This property shows for instance that the only bounded strictly pseudoconvex domain where h_D is a constant is the unit ball.

We will apply the properties of these functions to several questions. The first one is the following.

Let G be a bounded domain in $\mathbb{C}^n$, X a manifold. We will say that X can be exhausted by biholomorphic images of G (or shorter “by G ”) if for any compact $K \subset X$ there is a biholomorphic imbedding $F_K: G \rightarrow X$ such that $F_K(G) \supset K$.

Suppose that X is a hyperbolic manifold and X can be exhausted by G . The question is to describe X .

Questions similar to this one were studied in [1-4]. In [2] it was shown, in particular, that if a hyperbolic manifold X admits an exhaustion by an increasing sequence of biholomorphic images of the ball B^n (or U^n), then X is biholomorphically equivalent to B^n (or U^n). In this theorem B^n may be replaced by any bounded homogeneous domain (see Theorem 1.2 below). In [1] sharp estimates were obtained for the best imbedding of a polydisc in a ball and a ball in a polydisc. In terms of our functions these results are

$$h_{B^n}(x, U^n) = h_{U^n}(x, B^n) \equiv \left(\ln \left[\left(\sqrt{n+1} \right) / \left(\sqrt{n-1} \right) \right] \right)^{-1}.$$

In the direction of this question we obtain the following theorem.

THEOREM I. *Let $D \subset \mathbb{C}^n$ be a bounded strictly pseudoconvex domain with boundary of class C^3 , and X is a completely hyperbolic manifold of complex dimension n . Suppose it is possible to exhaust X by biholomorphic images of D . Then either X is biholomorphically equivalent to D or X is biholomorphically equivalent to B^n .*

As a corollary we obtain

THEOREM II. *Two bounded strictly pseudoconvex domains with boundaries of class C^3 are biholomorphically equivalent if and only if each of them can be exhausted by the other.*

To put it another way, if two such domains are “approximately” equivalent they are biholomorphically equivalent.

Theorems I and II were announced in [4].

Evidently any orbit of the group $\text{Aut}(D)$ of analytical automorphisms of a domain D lies on a level line of h_D . So h_D may give us some information about $\text{Aut}(D)$. We obtain a simple proof of the following theorem: if D is a bounded strictly pseudoconvex domain such that $\text{Aut}(D)$ is noncompact, then D is biholomorphically equivalent to a ball. This theorem is very well known for $\partial D \in C^\infty$ (see [11]). We prove it for $\partial D \in C^3$. One more generalization of this theorem can be found in §4. We prove there one theorem concerning $\text{Aut}(G)$ where G is a polyhedron.

In §1 we prove Theorem 1.2 about exhausting a hyperbolic manifold by a homogeneous domain. §2 is devoted to showing the relation between functions h_{X_n}

and h_X when X is exhausted by $\{X_n\}$. In §3 we prove property (0.2). In §4 we present the proofs of Theorems I, II, theorems concerning $\text{Aut}(D)$ and some remarks.

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1. Exhausting by biholomorphic images of a homogeneous domain. We begin with a general lemma.

LEMMA 1.1. *Let $D \subset \mathbb{C}^n$ be a bounded domain, X a hyperbolic manifold of complex dimension n . Suppose that there exist two relatively compact sets $K_1 \subset D$ and $K_2 \subset X$ and a sequence $\{F_k\}$ of mappings $F_k: D \rightarrow X$ that satisfy the following conditions:*

- (1) $F_k: D \rightarrow F_k(D)$ is biholomorphic for any $k < \infty$.
- (2) For any $1 \leq k < \infty$ there exists a point $z_k \in K_1$ such that $F_k(z_k) \in K_2$.
- (3) For any compact $K \subset X$ there exists a number $s = s(K)$ such that $F_s(D) \supset K$.

Then X is biholomorphically equivalent to D .

PROOF. (1) We denote $\Phi_k = F_k^{-1}: F_k(D) \rightarrow D$, $x_k = F_k(z_k)$. Let $U_1 \subseteq U_2 \subseteq \dots \subseteq U_n \subseteq \dots$ be a sequence of open sets such that $\bigcup U_n = X$. One can see that by passing to a subsequence if necessary we may assume that $\{\Phi_k\}$ satisfies the following condition: for each $n \geq 1$, there exists an N such that $\{\Phi_k\}_{k \geq N}$ is defined on U_n and converges uniformly on compacta in U_n to $\Phi: U_n \rightarrow \bar{D}$. Evidently, Φ is defined on X .

(2) Evidently $\Phi(X) \subset \bar{D}$. We want to show that $\Phi(X) \subset D$. Since $\{x_k\} \subset K_2$ and $\{z_k\} \subset K_1$, we may assume (by passing to a subsequence if necessary) that $\{x_k\}$ converges to $x_0 \in \bar{K}_2$ and $\{z_k\}$ to $z_0 \in \bar{K}_1$. One can easily see that $\Phi(x_0) = z_0$. Let $\varepsilon > 0$ be so small that the ball (in Kobayashi metric) $B_X(x_0, \varepsilon) \subseteq X$, and $B_D(z_0, \varepsilon) \subseteq D$. Let $z \in B_D(z_0, \varepsilon/2)$. For a large number k , $F_k(B_D(z_0, \varepsilon/2)) \subset B_X(x_0, \varepsilon)$ (we use here the important property of Kobayashi's metric: $\rho(F_k(x'), F_k(x'')) \leq \rho(x', x'')$). So, for this number k , $F_k(z) = x'_k \in B_X(x_0, \varepsilon)$. If x' is the limit point of $\{x'_k\}$ one can see that $\Phi(x') = z$. This means that $\Phi(X) \supset B_D(z_0, \varepsilon/2)$. Φ is a limit of regular holomorphic mappings. Since X is connected Φ may be either regular at any point at X or the Jacobian of Φ (in local coordinates) vanishes on X .

In the latter case $\Phi(X)$ could not contain any open set (by Sard's theorem). Since $\Phi(X) \supset B_D(z_0, \varepsilon/2)$, Φ is regular on X . This implies that $\Phi(X)$ is open, so $\Phi(X) \subset D$.

(3) We will show now that Φ is one-to-one. Let $x', x'' \in X$. For a large number k and the Kobayashi metric ρ we have

$$\begin{aligned} \rho(x', x'') &= \rho(F_k \circ \Phi_k(x'), F_k \circ \Phi_k(x'')) \leq \rho(\Phi_k(x'), \Phi_k(x'')) \\ &\leq \rho(\Phi_k(x'), \Phi(x')) + \rho(\Phi(x'), \Phi(x'')) + \rho(\Phi(x''), \Phi_k(x'')). \end{aligned}$$

When $k \rightarrow \infty$ we obtain

$$\rho(x', x'') \leq \rho(\Phi(x'), \Phi(x'')).$$

Hence, if $\Phi(x') = \Phi(x'')$, then $x' = x''$.

(4) We will show that $\Phi(X) \supset D$. From all we have already proved, X is biholomorphically equivalent to a domain $\Phi(X) \subset D$. From this point on we can

consider X as a bounded domain in $\mathbb{C}^n$. We may now assume that $\{F_k\}$ converges uniformly on compacta to $F: D \rightarrow \bar{X}$. It follows as in steps (1)–(3) that $F(z_0) = x_0$ and $F(D) \subset X$. For the mapping $\Phi \circ F: D \rightarrow D$ and any $z \in D$ we have

$$\Phi \circ F(z) = \lim_{k \rightarrow \infty} \Phi_k \circ F_k(z) = z.$$

Hence $\Phi(X) \supset D$.

It follows from (1)–(4) that $\Phi: X \rightarrow D$ is a biholomorphism, completing the proof of the lemma.

THEOREM 1.2. *Let $\mathcal{H} \subset \mathbb{C}^n$ be a bounded homogeneous domain, and let X be a hyperbolic manifold of complex dimension n . If X can be exhausted by biholomorphic images of $\mathcal{H}$, then X is biholomorphically equivalent to $\mathcal{H}$.*

PROOF. Let $\Phi_k: \mathcal{H} \rightarrow X$ be a sequence of biholomorphic mappings such that for any compact $K \subset X$ there exists a number s such that $\Phi_s(\mathcal{H}) \supset K$. Let $x_0 \in X$ and $z_0 \in \mathcal{H}$. Without loss of generality, we may suppose that $x_0 \in \Phi_k(\mathcal{H})$ for any k , $1 \leq k < \infty$. Since our domain is homogeneous, we can find $\psi_k \in \text{Aut}(\mathcal{H})$ such that $\psi_k(\Phi_k^{-1}(x_0)) = z_0$. Now the sequence $F_k = \Phi_k \circ \psi_k^{-1}$ satisfies all the conditions of Lemma 1.1 for $K_1 = \{z_0\}$ and $K_2 = \{x_0\}$. Applying this lemma we find that X is biholomorphically equivalent to $\mathcal{H}$.

THEOREM 1.3. *Let X be a connected hyperbolic manifold, $\dim_{\mathbb{C}} X = n$, and let $\mathcal{H} \subset \mathbb{C}^n$ be a bounded homogeneous domain. Then*

- (1) $h_X(x) = h_X(x, \mathcal{H})$ is continuous on X and $h_X \geq 0$.
- (2) If there is an $x_0 \in X$ such that $h_X(x_0) = 0$, then $h_X(x) \equiv 0$ and X is biholomorphically equivalent to $\mathcal{H}$.

PROOF. (1) If there exists an $x_0 \in X$ such that $h_X(x_0) = 0$ by Definition 0.1, there exists a sequence $\{\Phi_k\}$, $\Phi_k: \mathcal{H} \rightarrow X$, such that $\Phi_k(\mathcal{H}) \supset B_X(x_0, k)$. Now applying Theorem 1.2 we obtain that X is biholomorphically equivalent to $\mathcal{H}$ and, therefore, $h_X \equiv 0$.

(2) Let $h_X(x_0) > 0$ at any $x_0 \in X$. Let us consider $H_X(x_0) = 1/h_X(x) = \sup_{r \in \mathbb{R}} r$ (see Definition 0.1). It is enough to prove that H_X is continuous. Let $x_1, x_2 \in X$ be so close that

$$2\rho(x_1, x_2) < H_X(x_1).$$

Let $\varepsilon > 0$ be any number such that

$$H_X(x_1) - 2\rho(x_1, x_2) - 2\varepsilon > 0.$$

Since ρ is a metric, one can obtain

$$B_X(x_1, H_X(x_1) - \varepsilon) \supset B_X(x_2, H_X(x_1) - \varepsilon - \rho(x_1, x_2)).$$

This means that

$$H_X(x_2) \geq H_X(x_1) - \varepsilon - \rho(x_1, x_2).$$

So

$$(1.1) \quad H_X(x_1) - H_X(x_2) \leq \rho(x_1, x_2) + \varepsilon$$

and

$$H_X(x_2) - \varepsilon - \rho(x_1, x_2) > 0.$$

Similarly

$$B_X(x_2, H_X(x_2) - \varepsilon) \supset B_X(x_1, H_X(x_2) - \varepsilon - \rho(x_1, x_2))$$

and

$$(1.2) \quad H_X(x_2) - H_X(x_1) \leq \rho(x_1, x_2) + \varepsilon.$$

From (1.1) and (1.2), letting $\varepsilon \rightarrow 0$ yields

$$|H_X(x_2) - H_X(x_1)| \leq \rho(x_1, x_2).$$

This inequality shows that H_X is continuous in the topology induced by the Kobayashi metric. Since this topology is the usual one, we have finished the proof of Theorem 1.3.

2. Exhaustion and the behaviour of h_X . The main theorem we will prove in this section is the following one. (h_X below means $h_X(x, B^n)$.)

THEOREM 2.1. *Let X be a completely hyperbolic manifold and let $\{X_n\}$, $1 \leq n < \infty$, be a sequence of subdomains that exhausts X . Then, for any $x_0 \in X$,*

$$\lim_{n \rightarrow \infty} h_{X_n}(x_0) = h_X(x_0).$$

REMARK. By Definition 0.2 there exists an N such that $x_0 \in X_n$ for any $n \geq N$. The limit in the statement above is taken to mean $n \rightarrow \infty$ for $n \geq N$.

Before proving Theorem 2.1, we recall the definition of Kobayashi's metric [9].

Let M be a complex manifold, $p, q \in M$. Chain σ is a set of

- (1) m holomorphic mappings $f_i: B^1 \rightarrow M$, $f_i(B^1) \subseteq M$, $i = 1, \dots, m$, and
- (2) m pairs of points $z_j, w_j \in B^1$ such that $f_1(z_1) = p$, $f_m(w_m) = q$, and $f_j(w_j) = f_{j+1}(z_{j+1})$ for $j = 1, \dots, m-1$. Let $d(z, w)$ denote the usual Lobachevsky-Poincaré distance between z and w .

DEFINITION 2.2.

$$\rho_M(p, q) = \inf_{\sigma} \sum_{j=1}^m d(z_j, w_j)$$

where the inf is taken for all chains (m may be any number.)

The next property of this metric is particularly important. Let M_1 and M_2 be hyperbolic manifolds. If $F: M_1 \rightarrow M_2$ is holomorphic, then

$$(2.1) \quad \rho_{M_2}(F(x_1), F(x_2)) \leq \rho_{M_1}(x_1, x_2).$$

As a corollary we obtain

$$(2.2) \text{ If } M_1 \subset M_2, \text{ then } \rho_{M_2}(x_1, x_2) \leq \rho_{M_1}(x_1, x_2) \text{ for } x_1, x_2 \in M_1.$$

If $B_M(x_0, R) = \{x \mid \rho_M(x_0, x) < R\}$ is a ball in the Kobayashi metric on M , then (2.2) is equivalent to the following statement: for any $x_0 \in M_1$ and $R > 0$,

$$(2.3) \quad M_1 \subset M_2 \Rightarrow B_{M_1}(x_0, R) \subseteq B_{M_2}(x_0, R).$$

LEMMA 2.3. Let X be any complete hyperbolic manifold and let $\{X_n\}_{n=1}^\infty$ be a sequence of submanifolds that exhausts X . Let $x_0 \in X$ and $R > 0$. Then for any $\varepsilon > 0$ there exists an N such that $B_{X_n}(x_0, R) \supset B_X(x_0, R - \varepsilon)$ for all $n > N$.

PROOF. Let $\{U_n\}_{n=1}^\infty$ be a sequence of open submanifolds in X such that

(i) $U_{n+1} \supset U_n$ for all n ,

(ii) $\bigcup_{n=1}^\infty U_n = X$.

(1) Let x be a fixed element of X such that $\rho_X(x_0, x) \leq R - \varepsilon$. By Definition 2.2 there exists a chain $\sigma_m = \{f_i, z_i, w_i \mid i = 1, \dots, m\}$ such that

$$\sum_{i=1}^m d(z_i, w_i) \leq R - \varepsilon/2.$$

Since $\bigcup_{i=1}^m f_i(B^1) \subseteq X$, we can find a number l such that

$$U_l \supset \bigcup_{k=1}^m f_k(B^1).$$

By Definition 2.2,

$$\rho_{U_l}(x_0, x) \leq R - \varepsilon/2 < R.$$

Thus for any point $x \in X$ satisfying $\rho_X(x_0, x) \leq R - \varepsilon$, we have found an $l = l(x)$ such that $x \in B_{U_l}(x_0, R)$. Since $\overline{B_X(x_0, R - \varepsilon)}$ is compact (because X is complete), we can find a finite number of points $\{x_i\}_{i=1}^s$ such that

$$\bigcup_{i=1}^s B_{U_{l(i)}}(x_0, R) \supset \overline{B_X(x_0, R - \varepsilon)},$$

where $l(i) = l(x_i)$. Since the sequence $\{U_n\}_{n=1}^\infty$ is increasing (2.3) implies that

$$(2.4) \quad B_{U_L}(x_0, R) \supset \overline{B_X(x_0, R - \varepsilon)},$$

where $L = \max_{1 \leq i \leq s} l(x_i)$. There exists a number N such that $X_n \supset U_L$ for $n > N$. It follows from (2.3) and (2.4) that

$$B_{X_n}(x_0, R) \supseteq B_{U_L}(x_0, R) \supset \overline{B_X(x_0, R - \varepsilon)}.$$

The lemma is proved.

PROOF OF THEOREM 2.1. (1) Let $h_X(x_0) < 1/r$. There exists a biholomorphic mapping $F: B^n \rightarrow X$ such that $F(B^n) \supset B_X(x_0, r)$.

For any $\varepsilon > 0$, there exists a $\delta > 0$ such that $F((1 - \delta)B^n) \supset B_X(x_0, r - \varepsilon)$. Since $F((1 - \delta)B^n) \subseteq X$, there exists a number N such that $X_K \supset F((1 - \delta)B^n)$ for any $K > N$. Now

$$X_K \supset F((1 - \delta)B^n) \supset B_X(x_0, r - \varepsilon) \supset B_{X_K}(x_0, r - \varepsilon),$$

so

$$h_{X_K}(x_0) \leq 1/(r - \varepsilon).$$

We have proved that

$$\limsup h_{X_K}(x_0) \leq 1/r,$$

whenever $h_X(x_0) < 1/r$. Therefore

$$(2.5) \quad \limsup h_{X_k}(x_0) \leq h_X(x_0).$$

(2) Let

$$(2.6) \quad \liminf h_{X_k}(x_0) < 1/R$$

for some $R > 0$. By Lemma 2.3, for any $\varepsilon > 0$ we can find an N such that

$$(2.7) \quad B_{X_k}(x_0, R) \supset B_X(x_0, R - \varepsilon)$$

for any $K > N$. Now using (2.6) we can find a number $l > N$ such that

$$h_{X_l}(x_0) < 1/R.$$

Hence there exists a biholomorphic mapping $F: B^n \rightarrow X_l$ such that $F(B^n) \supset B_{X_l}(x_0, R)$. Together with (2.7), this yields

$$F(B^n) \supset B_X(x_0, R - \varepsilon), \quad \text{so} \quad h_X(x_0) \leq 1/(R - \varepsilon).$$

Since $\varepsilon > 0$ was arbitrary,

$$h_X(x_0) \leq 1/R.$$

Comparing this and the choice of R in (2.6), we see that

$$(2.8) \quad h_X(x_0) \leq \liminf h_{X_k}(x_0).$$

(3) From (2.5) and (2.8),

$$\lim_{n \rightarrow \infty} h_{X_n}(x_0) = h_X(x_0).$$

The theorem is proved.

REMARK 1. From the proof of this theorem one can see that $\lim_{n \rightarrow \infty} h_{X_n} = h_X$ uniformly on compacta.

2. The proof can be repeated for $h_X(x, \mathcal{H})$, where $\mathcal{H}$ is any bounded homogeneous domain for which $(1 - \delta)\mathcal{H} \subseteq \mathcal{H}$ for any $\delta > 0$. For instance, if $\mathcal{H} = U^n$, the theorem is true.

3. Behaviour of h_D near the boundary of a strictly pseudoconvex domain. In this section $h_D(z)$ means $h_D(z, B^n)$. The main purpose of this section is to prove the following.

THEOREM 3.1. *Let D be a bounded strictly pseudoconvex domain, $\partial D \in C^3$. Then $\lim_{z \rightarrow \partial D} h_D(z) = 0$.*

The next example shows that the ball B^n may be exhausted by biholomorphic images of half of the ball. The idea of this example will be used to prove Theorem 3.1.

EXAMPLE 3.2. Let $D = B^n \cap \{z \mid \operatorname{Re} z_1 > 0\}$. Let $\{\lambda_k\}_{k=1}^\infty \subset D$ be any sequence of points with the properties $\lambda_k = \alpha_k e_1$ and $\lim_{k \rightarrow \infty} \lambda_k = e_1$, where $\alpha_k \in \mathbf{R}$, $|\alpha_k| < 1$, $e_1 = (1, 0, \dots, 0)$.

We will prove that there exists a sequence $\{F_k\}$, $1 \leq k < \infty$, of mappings $F_k: D \rightarrow B^n$ with the properties:

(i) $F_k: D \rightarrow F_k(D)$ is biholomorphic for any $k < \infty$.

(ii) $F_k(\lambda_k) = 0$ for any $k < \infty$.

(iii) $\{F_k(D)\}_{k=1}^\infty$ exhausts B^n .

For this purpose let us put $z' = (z_2, \dots, z_n)$, $z = (z_1, z')$, $w = (w_1, w')$ and

$$(3.1) \quad F_k: \begin{cases} w_1 = (z_1 - \alpha_k)/(1 - z_1 \alpha_k), \\ w' = \sqrt{1 - \alpha_k^2} z' / (1 - z_1 \alpha_k). \end{cases}$$

F_k is an automorphism of the unit ball carrying λ_k to 0. $F_k: D \rightarrow B^n$ and (i) and (ii) evidently hold. Now let $|w| \leq R < 1$. From (3.1),

$$F_k^{-1}: \begin{cases} z_1 = (w_1 + \alpha_k)/(1 + w_1 \alpha_k), \\ z' = \sqrt{1 - \alpha_k^2} w' / (1 + w_1 \alpha_k). \end{cases}$$

Now

$$(3.2) \quad |z|^2 = |z_1|^2 + |z'|^2 = \frac{|w_1|^2 + \alpha_k^2 + w_1 \alpha_k + \bar{w}_1 \alpha_k + (1 - \alpha_k^2) |w'|^2}{|1 + w_1 \alpha_k|^2} \\ = \frac{(\alpha_k^2 |w_1|^2 + w_1 \alpha_k + \bar{w}_1 \alpha_k + 1) - (1 - \alpha_k^2)(1 - |w_1|^2 - |w'|^2)}{|1 + w_1 \alpha_k|^2} \\ = 1 - (1 - \alpha_k^2)(1 - R^2)/|1 + w_1 \alpha_k|^2 < 1.$$

For $\operatorname{Re} z_1$, we have

$$1 - \operatorname{Re} z_1 = \operatorname{Re} \left(1 - \frac{w_1 + \alpha_k}{1 + w_1 \alpha_k} \right) = \operatorname{Re} \frac{1 + w_1 \alpha_k - w_1 - \alpha_k}{1 + w_1 \alpha_k} \\ = \operatorname{Re} \frac{(1 - \alpha_k)(1 - w_1)}{1 + w_1 \alpha_k} \leq \frac{(1 - \alpha_k)(1 + R)}{1 - R}.$$

Thus

$$(3.3) \quad \operatorname{Re} z_1 \geq 1 - (1 - \alpha_k)(1 + R)/(1 - R).$$

From (3.2) and (3.3) one can see that, if α_k is close enough to 1, then $z \in D$. This shows that, for a large number N , $F_k(D) \supset \{w \mid |w| < R\}$ if $k > N$. This proves (iii).

The idea of the proof of Theorem 3.1 is as follows. We notice that it is enough to show that for any sequence $\{\lambda_k\} \subset D$, if $\lambda_k \rightarrow z^0 \in \partial D$, then $\lim_{k \rightarrow \infty} h_D(\lambda_k) = 0$. First we reduce this statement to a similar one concerning $D_\delta = D \cap U_\delta$, $U_\delta = \{z \mid |z - z^0| < \delta\}$ where $\delta > 0$ is any positive number. Then using the idea of Example 3.2 we find a sequence of mappings $F_k: D_\delta \rightarrow B^n$ with the same properties. Using then Theorem 2.1 we prove the statement $\lim_{k \rightarrow \infty} h_{D_\delta}(\lambda_k) = 0$.

I. We will now use Royden's definition of the Kobayashi metric (for details, see [10]). Let p be a point on a complex manifold X and let v be a vector on the tangent space $T_p(X)$ of X . Consider all holomorphic mappings $f: B_R^1 \rightarrow X$ (B_R^1 is a disk in $\mathbb{C}^1$ of radius R) satisfying the conditions $f(0) = p, f'(0) = v$. Define the functional

$$(3.4) \quad \Phi(p, v) = \frac{1}{\sup \{R: \exists f \in \mathcal{O}(B_R^1, X), f(0) = p, f'(0) = v\}}.$$

The distance $\rho(x_1, x_2)$ can be obtained by the formula

$$\rho(x_1, x_2) = \inf \int_0^1 \Phi(\gamma(t), \gamma'(t)) dt,$$

where the inf is taken over all piecewise smooth curves $\gamma: [0; 1] \rightarrow X$ such that $\gamma(0) = x_1$ and $\gamma(1) = x_2$.

The following estimate is a lemma of H. L. Royden (Lemma 2 in [10], a proof is given in [6]). We will need it in the following form.

LEMMA 3.3. *Let $D \subset \mathbb{C}^n$ be any bounded domain, $z_0 \in \partial D$, U a neighbourhood of z_0 , and $d = U \cap D$. There exists $\delta > 0$ such that, for any point $p \in d \cap U_\delta$, $U_\delta = \{z \mid |z - z_0| < \delta\}$ and any $v \in T_p(D)$,*

$$\Phi_D(p, v) \geq C\Phi_d(p, v),$$

where $C > 0$ is a constant independent of p .

LEMMA 3.4. *Let $D \subset \mathbb{C}^n$ be a bounded strictly pseudoconvex domain with $\partial D \in C^2$, $z_0 \in \partial D$, and $d = U \cap D$, where $U \ni z_0$ is a neighbourhood of z_0 . If $\lim_{z \rightarrow z_0} h_d(z) = 0$, then $\lim_{z \rightarrow z_0} h_D(z) = 0$.*

PROOF. (1) Let $\lambda \in d \cap U_\delta$, where U_δ is as in Lemma 3.3. Let $B_d(\lambda, r)$ be a ball in the Kobayashi metric such that $B_d(\lambda, r) \subset U_\delta$, and let $z \in [D \setminus B_d(\lambda, r)] \cap U_\delta$. One can see from Lemma 3.3 and Royden's definition of the Kobayashi metric that $\rho_D(\lambda, z) \geq Cr$. Hence

$$(3.8) \quad B_D(\lambda, (C/2)r) \subset B_d(\lambda, r).$$

(2) Let $\{\lambda_k\} \subset D$ be any sequence such that $\lim_{k \rightarrow \infty} \lambda_k = z_0$. Without loss of generality we may assume that $\{\lambda_k\}_{k=1}^\infty \subset U_\delta$. The condition $\lim_{k \rightarrow \infty} h_d(\lambda_k) = 0$ shows that there exists a sequence $\{R_k\}$, where $R_k \rightarrow \infty$ as $k \rightarrow \infty$, such that each ball $B_d(\lambda_k, R_k)$ can be covered by the image of an appropriate biholomorphic imbedding $F_k: B^n \rightarrow d$. From the behaviour of the Kobayashi metric near the point z_0 (for details see [6]) one can deduce that there exists a sequence $\{r_k\}$ such that $r_k \leq R_k$, $r_k \rightarrow \infty$, and $B_d(\lambda_k, r_k) \subset U_\delta$. Now, for each $k = 1, 2, \dots$, (3.8) implies that

$$B_D(\lambda_k, (C/2)r_k) \subset B_d(\lambda_k, r_k) \subset B_d(\lambda_k, R_k),$$

and therefore

$$h_D(\lambda_k) \leq (2/C)(1/r_k).$$

Since $r_k \rightarrow \infty$, we have proved that $\lim_{k \rightarrow \infty} h_D(\lambda_k) = 0$. Since the last statement holds for any $\{\lambda_k\}$ such that $\lim_{k \rightarrow \infty} \lambda_k = z_0$, we have proved Lemma 3.4.

II. D now means a fixed strictly pseudoconvex domain, $\bar{D} = \{z \mid \psi(z) \leq 0\}$, where $\psi(z)$ is a strictly plurisubharmonic function of class C^3 in a neighbourhood of $\bar{D}$ with $d\psi \neq 0$ on ∂D . Let $z^0 \in \partial D$, and let U be a neighbourhood of z^0 .

(1) First let T be a linear orthogonal transformation which carries z^0 onto the origin. We will denote $T(D)$ by D , $T(U)$ by U , and $\psi \circ T^{-1}$ by ψ .

If U is small enough, we have the following Taylor formula for ψ in U :

$$(3.9) \quad \psi = 2 \operatorname{Re} \sum_{\nu=1}^n \frac{\partial \psi}{\partial z_{\nu}} \bigg|_0 z_{\nu} + \operatorname{Re} \sum_{\mu, \nu=1}^n \frac{\partial^2 \psi}{\partial z_{\mu} \partial z_{\nu}} \bigg|_0 z_{\mu} z_{\nu} + \frac{1}{2} H_0(\psi, z) + O(|z|^3),$$

where $\frac{1}{2} H_0(\psi, z)$ is the Levy form.

Let us now introduce new coordinates

$$(3.10) \quad w_1 = \sum_{\nu=1}^n \frac{\partial \psi}{\partial z_{\nu}} \bigg|_0 z_{\nu} + \frac{1}{2} \sum_{\mu, \nu=1}^n \frac{\partial^2 \psi}{\partial z_{\mu} \partial z_{\nu}} \bigg|_0 z_{\mu} z_{\nu}$$

and $w' = (w_2, \dots, w_n)$ where $w_2, \dots, w_n$ are coordinates in the tangent space $T_0(\partial D)$ at 0. If U is small enough, this transformation is biholomorphic. In new coordinates our function (which is denoted now as $\psi_1(w)$) has the following representation:

$$\psi_1(w) = 2 \operatorname{Re} w_1 + \frac{1}{2} H_0(\psi_1, w) + O(|w|^3),$$

where $H_0(\psi, w)$ is the Levy form.

(2) Now we can find a linear transformation (we again use the letters z_{ν} for new coordinates) $z_1 = w_1$ and $z' = l(w')$ such that in these coordinates our function $\phi(z) = \psi_1(z_1, l^{-1}(z'))$ has the following representation:

$$(3.11) \quad \phi = 2 \operatorname{Re} z_1 + 2 \operatorname{Re}[z_1 L(\bar{z})] + |z'|^2 + R(z),$$

where L is a linear function, $z' = (z_2, \dots, z_n)$, and

$$(3.12) \quad |R(z)| \leq C_1 |z|^3, \quad C_1 = \text{const.}$$

We define a domain D_{δ} as the set of solutions of the following system:

$$(3.13) \quad \begin{cases} 2 \operatorname{Re} z_1 + 2 \operatorname{Re}[z_1 L(\bar{z})] + |z'|^2 + R(z) < 0, \\ |z|^2 < \delta. \end{cases}$$

Since ϕ is strictly pseudoconvex,

$$2 \operatorname{Re}[z_1 L(\bar{z})] + |z'|^2 \geq \sigma |z|^2, \quad \sigma > 0.$$

Taking (3.12) into account one can see that, for small enough $\delta > 0$,

$$(3.14) \quad |z|^2 \leq C_2 (-2 \operatorname{Re} z_1) \quad \text{for any } z \in D_{\delta}$$

where $C_2 = \text{const} > 0$. We assume that such a $\delta > 0$ is chosen and fixed.

(3) Let $\varepsilon > 0$, and consider the transformation $\Phi_{\varepsilon}: D_{\delta} \rightarrow \mathbb{C}^n$ described by

$$(3.15) \quad \Phi_{\varepsilon}: \begin{cases} w_1 = \frac{2z_1 + \varepsilon}{2z_1 - \varepsilon}, \\ w' = \frac{2\sqrt{\varepsilon} z'}{2z_1 - \varepsilon}; \end{cases} \quad \Phi_{\varepsilon}^{-1}: \begin{cases} z_1 = \varepsilon \frac{w_1 + 1}{2(w_1 - 1)}, \\ z' = \sqrt{\varepsilon} \frac{w'}{w_1 - 1}. \end{cases}$$

$\psi(w) = \Phi(\Phi_\varepsilon^{-1}(w))$ will have the following representation:

$$(3.16) \quad \psi(w) = \varepsilon \frac{(w_1 + 1)(\bar{w}_1 - 1) + (\bar{w}_1 + 1)(w_1 - 1)}{2|w_1 - 1|^2} + 2 \operatorname{Re} \left| \varepsilon \frac{w_1 + 1}{2(w_1 - 1)} \frac{\sqrt{\varepsilon}}{\bar{w}_1 - 1} L_1(\bar{w}) \right| + \varepsilon \frac{|w'|^2}{|w_1 - 1|^2} + R(\Phi_\varepsilon^{-1}(w))$$

$$= \varepsilon \frac{|w_1|^2 - 1}{|w_1 - 1|^2} + \varepsilon \frac{|w'|^2}{|w_1 - 1|^2} + \varepsilon \sqrt{\varepsilon} \frac{s(w, \bar{w})}{|w_1 - 1|^2} + R(\Phi_\varepsilon^{-1}(w)),$$

where $s(w, \bar{w})$ is a polynomial. The representation of $\Phi_\varepsilon(D_\delta)$ can now be obtained from (3.13) and (3.16):

$$(3.17) \quad \begin{cases} |w|^2 - 1 + \sqrt{\varepsilon} s(w_1, \bar{w}) + (1/\varepsilon) R(\Phi_\varepsilon^{-1}(w)) |w_1 - 1|^2 < 0, \\ 4\delta |w_1 - 1|^2 > \varepsilon(\varepsilon |w_1 + 1|^2 + 4|w'|^2). \end{cases}$$

(4) We will need below the statement: for any $\varepsilon > 0$ small enough,

$$(3.18) \quad |w| = |\Phi_\varepsilon(z)| < C_3 = \text{const} \quad \text{for any } z \in D_\delta.$$

To prove this we use the representation (3.15):

$$(3.19) \quad |w_1| = \left| 1 + \frac{2\varepsilon}{2z_1 - \varepsilon} \right| \leq 1 + \frac{2\varepsilon}{|2 \operatorname{Re} z_1 - \varepsilon|} \leq 1 + \frac{2\varepsilon}{0 + \varepsilon} = 3,$$

$$(3.20) \quad |w'| = \frac{2\sqrt{\varepsilon} |z'|}{|2z_1 - \varepsilon|} \leq \frac{2\sqrt{\varepsilon} \sqrt{C_2(-2 \operatorname{Re} z_1)}}{(-2 \operatorname{Re} z_1) + \varepsilon} \leq \sqrt{C_2}.$$

(Here we have used the inequality $2 \operatorname{Re} z_1 < 0$ and (3.14).) From (3.19) and (3.20) we get (3.18).

(5) Next we prove that

$$(3.21) \quad |(1/\varepsilon) R(\Phi_\varepsilon^{-1}(w))| |w_1 - 1|^2 \leq C_4 \sqrt{\varepsilon}.$$

Using (3.15), (3.12) and (3.14), we obtain

$$\begin{aligned} \left| \frac{1}{\varepsilon} R(\Phi_\varepsilon^{-1}(w)) |w_1 - 1|^2 \right| &\leq \frac{1}{\varepsilon} C_1 |z|^3 \left| \frac{2\varepsilon}{2 \operatorname{Re} z_1 - \varepsilon} \right|^2 \\ &\leq 4C_1 C_2^{3/2} \frac{(-2 \operatorname{Re} z_1)^{3/2} \varepsilon}{(-2 \operatorname{Re} z_1 + \varepsilon)^2} \leq C_4 \sqrt{\varepsilon}. \end{aligned}$$

The last inequality can easily be obtained by finding that the last but one expression has its maximum (for $-2 \operatorname{Re} z_1 > 0$) when $-2 \operatorname{Re} z_1 = 3\varepsilon$.

(6) $\Phi_\varepsilon(D_\delta)$ now has the following description (from (3.17)):

$$(3.22) \quad \begin{cases} |w|^2 - 1 + \sqrt{\varepsilon} A(w) < 0, \\ 4\delta |w_1 - 1|^2 > \varepsilon(\varepsilon |w_1 + 1|^2 + 4|w'|^2), \end{cases}$$

where $|A(w)| \leq C_5 = \text{const}$. From (3.18) it follows that

$$(3.23) \quad \Phi_\varepsilon(D_\delta) \rightarrow B^n \quad \text{as } \varepsilon \rightarrow 0.$$

The first limit in (3.23) is understood as a limit of sets.

(7) Let us return now to the initial domain D . Let $\lambda_k \rightarrow z^0 \in \partial D$. For each λ_k we choose $z^k \in \partial D$ to be one of the points on ∂D closest to λ_k . Without loss of generality we may assume that for each λ_k such a point z^k is unique. We can now change the coordinate system so that the line joining z^k and λ^k is the $\text{Re } z_1$ -axis and z^k is the origin. Then we can obtain the representation (3.11). Let the image of λ_k after all these transformations be $-2\varepsilon_k e_1$ where $\varepsilon_k > 0$. Now we can consider the transformation $\Phi_{2\varepsilon_k}$ (see (3.15)). One can see that all the constants δ , C_1 – C_5 are uniformly bounded for all $k \geq N$ for some N . Without loss of generality we may assume that it holds for all k . Now from (3.23),

$$(3.24) \quad \Phi_{2\varepsilon_k}(D_\delta) \rightarrow B^n \text{ in the sense of a limit of sets.}$$

From (3.15) with ε replaced by $2\varepsilon_k$,

$$(3.25) \quad \Phi_{2\varepsilon_k}(\lambda_k) = 0.$$

(8) We have obtained the following. There exist a neighbourhood $U \ni z^0$ and a sequence $\{F_k\}$ of biholomorphic transformations of $d = D \cap U$ such that

$$(3.26) \quad F_k(\lambda_k) = 0,$$

$$(3.27) \quad F_k(d) \rightarrow B^n \text{ uniformly in the sense of the limit of sets.}$$

(9) It is clear from (3.27) that for each k there exists a number $\delta_k > 0$, $\lim_{k \rightarrow \infty} \delta_k = 0$, such that if we set $T_k = (1 - \delta_k)F_k$ then

$$(i) \quad T_k(d) \subset B^n \text{ for each } k,$$

$$(ii) \quad \{T_k(d)\} \text{ exhausts } B^n.$$

Now (3.26) implies that $T_k(\lambda_k) = 0$, and Theorem 2.1 yields

$$(3.28) \quad \lim_{k \rightarrow \infty} h_d(\lambda_k) = \lim_{k \rightarrow \infty} h_{T_k(d)}(0) = h_{B^n}(0) = 0.$$

(10) Theorem 3.1 now follows from (3.28) and Lemma 3.4. This completes the proof of Theorem 3.1.

4. Proofs of Theorems I and II, and remarks.

1. PROOF OF THEOREM I. Let $F_n: D \rightarrow X$, $n = 1, 2, \dots$, be a sequence of biholomorphic imbeddings such that, for any compact $K \subset X$, there exists N such that $F_n(D) \supset K$ for all $n > N$. The existence of such a sequence follows from the conditions of the theorem. Let $x_0 \in X$. Without loss of generality we may assume that $F_n(D) \ni x_0$ for every n . We now consider the set $A = \{F_n^{-1}(x_0)\} \subset D$. There are two cases to be considered.

Case 1. A is relatively compact in D . Then by Lemma 1.1 X is biholomorphically equivalent to D .

Case 2. A has at least one limit point $z^0 \in \partial D$. Without loss of generality (by passing to a subsequence if necessary) we may suppose that $\lim_{n \rightarrow \infty} F_n^{-1}(x_0) = z^0$. From Theorem 3.1 it follows that $\lim_{n \rightarrow \infty} h_D(F_n^{-1}(x_0)) = 0$. Now Theorem 2.1 and the fact that h is invariant imply that

$$h_X(x_0) = \lim_{n \rightarrow \infty} h_{F_n(D)}(x_0) = \lim_{n \rightarrow \infty} h_D(F_n^{-1}(x_0)) = 0.$$

Theorem 1.3 (for $\mathcal{K} = B^n$) now shows that X is biholomorphically equivalent to B^n .

2. PROOF OF THEOREM II. Let D_1 and D_2 be the two pseudoconvex domains. Since D_2 can be exhausted by biholomorphic images of D_1 it follows from Theorem I that either D_2 is biholomorphically equivalent to D_1 , or D_2 is biholomorphically equivalent to B^n . In the latter case, since D_1 can be exhausted by D_2 , it follows from Theorem I that D_1 is biholomorphically equivalent to B^n and, therefore, to D_2 . Theorem II is proved.

Let $D \subset \mathbb{C}^n$ be a bounded domain. Let $z^0 \in \partial D$ have a neighbourhood U such that $\partial D \cap U$ is strictly pseudoconvex and of class C^3 . It follows as in the proof of Theorem 3.1 that

$$(4.1) \quad \lim_{z \rightarrow z^0} h_D(z) = 0.$$

(In fact this very statement was proved in section 3; it follows from Lemma 3.4 and (3.28) for an appropriate $d \subset D \cap U$.)

We now denote by $\text{Aut}(D)$ the group of analytical automorphisms of D . If $z' \in D$, $\Gamma(z') = \{z \mid z = F(z'), F \in \text{Aut}(D)\}$ is the orbit of z' . If we assume that the closure $\overline{\Gamma}(z')$ contains z^0 , then there exists a sequence $\{F_k\} \subset \text{Aut}(D)$ such that $\lim_{k \rightarrow \infty} F_k(z') = z^0$. Now using (4.1) we obtain

$$h_D(z^0) = \lim_{k \rightarrow \infty} h_D(F_k(z')) = 0.$$

Hence it follows from Theorem 1.3 that D is biholomorphically equivalent to B^n . We have proved the following statement.

LEMMA 4.1. *Let $D \subset \mathbb{C}^n$ be a bounded domain. Let $z^0 \in \partial D$ have a neighbourhood U such that $\partial D \cap U$ is strictly pseudoconvex and belongs to the class C^3 . If there exists a point $z' \in D$ such that the closure of its orbit $\overline{\Gamma}(z')$ contains z^0 , then D is biholomorphically equivalent to B^n .*

As a corollary of Lemma 4.1 we obtain a new proof of the following well-known theorem (for $\partial D \in C^\infty$, see [11]):

THEOREM 4.2. *Let D be a bounded strictly pseudoconvex domain with $\partial D \in C^3$. If $\text{Aut}(D)$ is noncompact, then D is biholomorphically equivalent to B^n .*

More information about $\text{Aut}(D)$ can be found in [7].

If G is a bounded domain, $z^0 \in \partial G$, and z^0 has a neighbourhood on ∂G which is similar to a part of the boundary of the unit polydisk U^n , we can prove an analogue of Theorem 3.1. More precisely, let there be an analytic function $f(z)$ in a neighbourhood U of z^0 such that

$$(4.2) \quad \partial G \cap U = \{z \mid |f(z)| = 1\}.$$

Using the functions $h_G(x, U^n)$ in place of the $h_G(x, B^n)$ we find that $\lim_{z \rightarrow z^0} h_G(z, U^n) = 0$. Analogously to Lemma 4.1 one can obtain

LEMMA 4.3. *Let $D \subset \mathbb{C}^n$ be a bounded domain. Let $z^0 \in \partial G$ have a neighbourhood U with property (4.2) (for some analytic function f). If for some point $z' \in G$ the closure of the orbit (with respect to $\text{Aut } G$) of z' contains z^0 , then G is biholomorphically equivalent to U^n .*

Now let $G \subset \mathbb{C}^n$ be a bounded polyhedron. Then G is a bounded domain such that $G = \{z \in \mathbb{C}^n \mid |f_s| < 1, s = 1, \dots, p\}$, where the functions $f_1, \dots, f_p$ are holomorphic in a neighbourhood of the closure of G . Then the $(2n-1)$ -dimensional measure of all points having no neighbourhood U with property (4.2) is zero. Let

$$M(G) = \bigcup_{z \in G} \overline{\Gamma(z)} \cap \partial G$$

and let $\mu_{2n-1}(M)$ be the $(2n-1)$ -dimensional measure of M . If G is the polydisk U^n , then evidently $M(G) = \partial U^n$.

Lemma 4.3 yields

THEOREM 4.4. *If G is a bounded polyhedron and $\mu_{2n-1}(M(G)) > 0$ then G is biholomorphically equivalent to U^n .*

Biholomorphic classification of polyhedra and $\text{Aut } G$ were considered in [5].

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